A_∞ ALGEBRAS AND THE
COHOMOLOGY OF MODULI SPACES

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§1. INTRODUCTION

Let us consider an A_∞ algebra with an invariant inner product. The main goal of this paper is to classify the infinitesimal deformations of this A_∞ algebra preserving the inner product and to apply this result to the construction of homology classes on the moduli spaces of algebraic curves. With this aim, we define cyclic cohomology of an A_∞ algebra and show that it classifies the deformations we are interested in. To make the reading of our paper more independent of other works, we include a short review of Hochschild and cyclic cohomology of associative algebras, and explain the definition of A_∞ algebras.

Our constructions are based on ideas and results of Maxim Kontsevich; moreover, he has informed us that he also has given a definition of the cyclic cohomology of A_∞ algebras in a different manner than we do and has proved the results mentioned above as well. Another definition of cyclic cohomology of A_∞ algebras was given by Ezra Getzler and John D.S. Jones [5]. We did not study its relation to our definition.

In this paper we make the notational convention when dealing with $\mathbb{Z}_2$ grading, that if a is a homogeneous element with parity $|a|$, in superscripts we will use a in place of $|a|$, so that $(-1)^a$ stands for $(-1)^{|a|}$, and similarly, $(-1)^{ab} = (-1)^{|a||b|}$, not $(-1)^{|ab|}$, which is of course given by $(-1)^{a+b}$.

§2. COHOMOLOGY OF ASSOCIATIVE ALGEBRAS

In this section, we recall the definition of Hochschild cohomology of associative algebras. We relate this notion to the theory of deformations of the associative algebra structure. Then we discuss the theory of deformations of an associative algebra preserving an invariant inner product, and relate this notion to cyclic cohomology. The purpose of this is to motivate the definition of A_∞ algebras, and to set the stage for the more general discussion of cohomology of A_∞ algebras.

In this section, suppose that A is an algebra over a field k. For simplicity, we suppose that A is finite-dimensional over k. Let $C^n(A) = \operatorname{Hom}(A^n, A)$ be the space of n-multilinear functions on A with values in A; we call $C^n(A)$ the module

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of cochains of degree n on A with values in A. We define a coboundary operator $b : C^n(A) \to C^{n+1}(A)$ by

$$
\begin{aligned}
bf(a_1, \ldots, a_{n+1}) =\; & a_1 f(a_2, \ldots, a_{n+1}) \\
& + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\
& + (-1)^{n+1} f(a_1, \ldots, a_{n+1})
\end{aligned}
\tag{1}
$$

Then

$$
HH^n(A) = \ker(b : C^n(A) \to C^{n+1}(A)) / \operatorname{im}(b : C^{n-1}(A) \to C^n(A))
$$

is the Hochschild cohomology of A with coefficients in A. In this paper we do not consider cohomology with other coefficients. The connection between Hochschild cohomology and (infinitesimal) deformations of A is given by $HH^2(A)$. If we denote the product in A by m, an infinitesimally deformed product by m_t, and express $m_t = m + t\phi$, where $t^2 = 0$, then the map $\varphi : A^2 \to A$ is a Hochschild cocycle and the trivial deformations are given by Hochschild coboundaries. To show the first assertion note that

$$
\begin{aligned}
m_t(m_t(a_1, a_2), a_3)) =\; & a_1 a_2 a_3 + t(\varphi(a_1, a_2)a_3 + \varphi(a_1 a_2, a_3)), \\
m_t(a_1, m_t(a_2, a_3)) =\; & a_1 a_2 a_3 + t(a_1 \varphi(a_2, a_3) + \varphi(a_1, a_2 a_3)).
\end{aligned}
$$

Associativity of m_t is equivalent to the condition

$$
a_1 \varphi(a_2, a_3) - \varphi(a_1 a_2, a_3) + \varphi(a_1, a_2 a_3) - \varphi(a_1, a_2)a_3 = 0.
\tag{2}
$$

But this last condition is simply the condition $b\varphi = 0$.

On the other hand, the notion of a trivial deformation is given by the condition that A with the new multiplication is isomorphic to the original algebra structure. This means that there is a linear bijection $\rho_t : A \to A$ such that $m_t(\rho_t(a_1), \rho_t(a_2)) = \rho_t(a_1 a_2)$. We can express $\rho_t = I + t\lambda$, where $\lambda : A \to A$ is a linear map. Then

$$
\begin{aligned}
m_t(a_1, a_2) &= \rho_t(\rho_t^{-1}(a_1), \rho_t^{-1}(a_2)) \\
&= a_1 a_2 - t(a_1 \lambda(a_2) - \lambda(a_1 a_2) + \lambda(a_1)a_2) = a_1 a_2 - t(b\lambda)(a_1, a_2).
\end{aligned}
$$

Thus coboundaries give rise to trivial deformations.

Next, consider an invariant non-degenerate inner product on A, by which we mean an inner product $\langle \cdot, \cdot \rangle$ that satisfies $\langle ab, c \rangle = \langle a, bc \rangle$. Note that for an invariant inner product, we also have that $\langle ab, c \rangle = \langle ca, b \rangle$, so that it is invariant under the cyclic permutations of a, b, c. We consider deformations of A preserving this inner product, and these are governed by cyclic cohomology. To see this connection we first note that in the presence of an inner product, there is a natural isomorphism $\operatorname{Hom}(A^n, A) \xrightarrow{\sim} \operatorname{Hom}(A^{n+1}, k)$, given as follows. If we denote the image of $f \in \operatorname{Hom}(A^n, A)$ in $\operatorname{Hom}(A^{n+1}, k)$ by $\widetilde{f}$, then

$$
\widetilde{f}(a_1, \ldots, a_{n+1}) = \langle f(a_1, \ldots, a_n), a_{n+1} \rangle.
$$

If we define an element $\widetilde{f}$ in $\mathrm{Hom}(A^n, k)$ to be cyclic whenever

$$\widetilde{f}(a_1, \ldots, a_n) = (-1)^{n+1} \widetilde{f}(a_n, a_1, \ldots, a_{n-1}),$$

then we see that a cyclic element $\widetilde{\varphi}$ in $\mathrm{Hom}(A^3, k)$ corresponds to a deformation φ in $C^2(A)$ preserving the inner product, because

$$(3) \qquad \langle \varphi(a, b), c \rangle = \widetilde{\varphi}(a, b, c) = \widetilde{\varphi}(b, c, a) = \langle a, \varphi(b, c) \rangle$$

A trivial deformation preserving the inner product is determined by a linear map $\rho_t = I + t\lambda$ as before, but in addition we assume that

$$(4) \qquad \langle \rho_t(a_1), \rho_t(a_2) \rangle = \langle a_1, a_2 \rangle.$$

This is equivalent to the condition $\langle \lambda(a_1), a_2 \rangle = -\langle a_1, \lambda(a_2) \rangle$. This latter condition is precisely the condition that $\widetilde{\lambda}(a_1, a_2) = -\widetilde{\lambda}(a_2, a_1)$. In other words, λ is cyclic.

The Hochschild coboundary operator b induces a coboundary operator $\widetilde{b}$: $\mathrm{Hom}(A^{n+1}, k) \to \mathrm{Hom}(A^{n+2}, k)$ by $\widetilde{b}\widetilde{f} = \widetilde{bf}$. As we shall show later, the coboundary operator takes cyclic elements to cyclic elements. If we denote the submodule of $\mathrm{Hom}(A^{n+1}, k)$ consisting of cyclic elements by $CC^n(A)$, then the cyclic cohomology of A is defined by

$$HC^n(A) = \ker(\widetilde{b} : CC^n(A) \to CC^{n+1}(A)) / \mathrm{im}(\widetilde{b} : CC^{n-1}(A) \to CC^n(A))$$

This definition depends on the choice of the inner product. However, one can express the coboundary operator $\widetilde{b}$ without reference to the inner product. If $f \in \mathrm{Hom}(A^{n+1}, k)$, then we see that

$$
\begin{aligned}
\widetilde{b}\widetilde{f}(a_1, \ldots, a_{n+2}) &= \widetilde{bf}(a_1, \ldots, a_{n+2}) = \langle bf(a_1, \ldots, a_{n+1}), a_{n+2} \rangle \\
&= \langle a_1 f(a_2, \ldots, a_{n+1}), a_{n+2} \rangle + \sum_{i=1}^{n} (-1)^i \langle f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}), a_{n+2} \rangle \\
&\quad + (-1)^{n+1} \langle f(a_1, \ldots, a_n) a_{n+1}, a_{n+2} \rangle \\
&= \langle f(a_2, \ldots, a_{n+1}), a_{n+2} a_1 \rangle + \sum_{i=1}^{n} (-1)^i \langle f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}), a_{n+2} \rangle \\
&\quad + (-1)^{n+1} \langle f(a_1, \ldots, a_n), a_{n+1} a_{n+2} \rangle \\
&= \widetilde{f}(a_2, \ldots, a_{n+1}, a_{n+2} a_1) + \sum_{i=1}^{n+1} (-1)^i \widetilde{f}(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+2})
\end{aligned}
$$

(5)

If $\widetilde{f}$ is cyclic, then we see that

$$
\begin{aligned}
&\widetilde{b}\widetilde{f}(a_1, \ldots, a_{n+2}) \\
&\quad = (-1)^n \widetilde{f}(a_{n+2} a_1, a_2, \ldots, a_n) + \sum_{i=1}^{n+1} (-1)^{ni+n+1} \widetilde{f}(a_i a_{i+1}, \ldots, a_{i-1}) \\
&\quad = \sum_{i=0}^{n+1} (-1)^{n[i]+n+1} \widetilde{f}(a_{[i]} a_{[i+1]}, \ldots, a_{[i-1]}),
\end{aligned}
$$

(6)

where $[i] = i \pmod{n+2}$, and we make the convention that $a_0 = a_{n+2}$. The fact that $\widetilde{bf}$ is cyclic follows easily from this formula for the differential.

Thus the cyclic cohomology characterizes the deformations of A that preserve an invariant inner product, independently of the particular inner product involved. We note that cyclic homology is usually based on applying the operator b to the complex $C^n(A, A^*) = \mathrm{Hom}(A^n, A^*)$. However, the description we give here is equivalent to this one, because the inner product induces a natural isomorphism between $\mathrm{Hom}(A^n, A^*)$ and $\mathrm{Hom}(A^n, A)$. Of course, the description in terms of $C^n(A, A^*)$, being independent of any inner product, makes completely transparent the fact that the cyclic homology does not depend on the inner product. However, it does not elucidate the connection between $HC^2(A)$ and deformations preserving an inner product. As was pointed out to us by Getzler, this relation was first shown by Connes-Flato-Sternheimer in [1].

§3. Cohomology of $\mathbb{Z}_2$-graded algebras

Let us consider a $\mathbb{Z}_2$-graded algebra A over a ring k. The notion of an invariant inner product needs to be modified in this case. The definition of a graded symmetric inner product is given by the formula

$$\langle a, b \rangle = (-1)^{ab} \langle b, a \rangle$$

for homogeneous elements a, b in A. Let k be a commutative ring. Consider elements of k as having degree 0, making k a $\mathbb{Z}_2$-graded ring in a trivial sense. If we require the inner product to be an even map when considered as a map from $A \otimes A$ to k, then $\langle a, b \rangle = 0$ unless a and b have the same parity. On elements of even parity the inner product is symmetric, and on elements of odd parity the inner product is a skew symmetric form. This definition is for an even inner product. One can also consider the case when k is a supercommutative ring. Odd inner products can also be defined, but we will not consider them in this paper. The notion of an invariant inner product is given by the same relation $\langle ab, c \rangle = \langle a, bc \rangle$, but now we have

$$
\begin{aligned}
(7) \qquad \langle ab, c \rangle &= \langle a, bc \rangle = (-1)^{(b+c)a} \langle bc, a \rangle = (-1)^{(b+c)a} \langle b, ca \rangle \\
&= (-1)^{(b+c)a + (c+a)b} \langle ca, b \rangle = (-1)^{c(a+b)} \langle ca, b \rangle
\end{aligned}
$$

We also assume that the multiplication is an even map, so that $|ab| = |a| + |b|$. We are only interested in deformations of A that preserve this property. If we denote the deformed multiplication by $m_t = m + t\varphi$, and consider t as an even parameter, then φ must be even map, but if t is an odd parameter, then φ must be odd. For parameters, we assume the property of graded commutativity, so that $ta = (-1)^{ta} at$. The associativity condition (2) is modified by this consideration, so that now we have the formula

$$(8) \qquad (-1)^{a_1 \varphi} a_1 \varphi(a_2, a_3) - \varphi(a_1 a_2, a_3) + \varphi(a_1, a_2 a_3) - \varphi(a_1, a_2) a_3 = 0.$$

which is the deformation condition for any homogeneous bilinear map φ.

Now we consider a trivial deformation, which, as before, is given by a linear map $\rho_t = I + t\lambda : A \to A$, but the parameter is allowed to be odd, in which case λ is odd as well. In this case, a trivial deformation is given by

$$m_t(a_1, a_2) = a_1 a_2 - t(a_1 \lambda(a_2) - \lambda(a_1 a_2) + (-1)^{a_1} a_1 \lambda(a_2)).$$

These two results suggest that we should define the Hochschild coboundary operator in the $\mathbb{Z}_2$-graded case as the map $b : C^n(A) \to C^{n+1}(A)$ given by

$$
\begin{aligned}
bf(a_1, \ldots, a_{n+1}) = &(-1)^{a_1 f} a_1 f(a_2, \ldots, a_{n+1}) \\
&+ \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\
&+ (-1)^{n+1} f(a_1, \ldots, a_n) a_{n+1}
\end{aligned}
$$
(9)

for homogeneous f. It is easily checked that $b^2 = 0$. Then deformations of A correspond to cocycles, while trivial deformations correspond to coboundaries of this new Hochschild coboundary operator.

The definition of a cyclic element of $CC^n(A)$ is adjusted to be consistent with the grading, which makes the notion compatible with the definition of the invariance of the inner product. This is accomplished by defining an element $\widetilde{f} \in CC^n(A)$ to be cyclic if and only if

$$
(10) \qquad \widetilde{f}(a_1, \ldots, a_{n+1}) = (-1)^{n + a_{n+1}(a_1 + \cdots + a_n)} \widetilde{f}(a_{n+1}, a_1, \ldots, a_n).
$$

Note that if $\widetilde{m} \in CC^2(A)$ is given by $\widetilde{m}(a, b, c) = \langle ab, c \rangle$, then $\widetilde{m}$ is cyclic precisely when the inner product is invariant with respect to the multiplication. Also, we see that $\widetilde{f}$ is cyclic precisely when

$$
(11) \qquad \langle f(a_1, \ldots, a_n), a_{n+1} \rangle = (-1)^{n + a_1 f} \langle a_1, f(a_2, \ldots, a_{n+1}) \rangle
$$

for all homogeneous elements.

We want to express the Hochschild differential of cyclic homology in terms of this new notion of cyclicity. Let us restrict ourselves to the case when k is a field. Then, as in the nongraded case, there is a natural isomorphism between $\mathrm{Hom}(A^n, A)$ and $\mathrm{Hom}(A^{n+1}, k)$ defined in the same manner as before, inducing a coboundary operator $\widetilde{b} : \mathrm{Hom}(A^n, k) \to \mathrm{Hom}(A^{n+1}, A)$ given by $\widetilde{b}\widetilde{f} = \widetilde{bf}$. The proof of the formula below is straightforward, and cyclicity is easily seen from this formula.

Lemma. *The Hochschild differential $\widetilde{b}$ takes cyclic elements to cyclic elements. For a cyclic element $\widetilde{f}$, the Hochschild differential can be expressed in the form*

$$
\begin{aligned}
&\widetilde{b}\widetilde{f}(a_1, \ldots, a_{n+1}) \\
(12) \quad &= \sum_{i=0}^{n} -1^{ni + n + 1 + (a_1 + \cdots + a_{[i-1]})(a_{[i]} + \cdots + a_{n+1})} \widetilde{f}(a_{[i]} a_{[i+1]}, a_{[i+2]}, \ldots, a_{[i-1]}),
\end{aligned}
$$

where $[i] = i \pmod{n+1}$ and by convention, $a_0 = a_{n+1}$.

The sign in the expression above, as in the formulas below, follow from the exchange rule which is given by the principle that when two elements are exchanged in an expression, the corresponding sign is -1 to the power equal to the product of the parities of the two elements.

As in the nongraded case, one sees that the cyclic cohomology $HC^2(A)$ classifies the deformations of A which preserve a graded symmetric inner product. Cyclic cohomology of $\mathbb{Z}_2$-graded algebras was considered by D. Kastler in [8].

§4. Definition of A_∞ Algebras

Suppose that W is a $\mathbb{Z}_2$-graded space over k, and denote the tensor algebra over W by $T(W) = \bigoplus_{n=1}^{\infty} W^n$, where $W^1 = W$ and $W^{n+1} = W \otimes W^n$. Then $T(W)$ is an associative algebra under the tensor product. As usual, the parity of a product $w = a_1 \otimes \cdots \otimes a_n$ is given by $|w| = |a_1| + \cdots + |a_n|$.

In this picture, we let $\widehat{d}$ be a (super) derivation on $T(W)$; by this we mean that $|\widehat{d}\omega| = |\omega| + |d|$ for homogeneous $\omega \in T(W)$, where $|d|$ is the parity of the map d, and also $\widehat{d}(\omega \otimes \eta) = \widehat{d}(\omega) \otimes \eta + (-1)^{\omega d}\omega \otimes \widehat{d}\eta$, which is the derivation law for (super) derivations. If we denote the restriction of $\widehat{d}$ to W by d, then the derivation law shows that $\widehat{d}$ is uniquely determined by d. Denote $d = d_1 + d_2 + \cdots$, where $d_k : W \to W^k$ is the induced map, and similarly denote $\widehat{d} = \widehat{d}_1 + \widehat{d}_2 + \cdots$, where $\widehat{d}_l$ is the component of $\widehat{d}$ of degree $l - 1$ and where for each k, the restriction $d_{l,k}$ of $\widehat{d}_l$ to W^k is the map $d_{l,k} : W^k \to W^{k+l-1}$ induced by $\widehat{d}$. We can express $d_{l,k}$ in terms of d_l by

$$d_{l,k} = \sum_{i+j=k-1} I_i \otimes d_l \otimes I_j$$

where $I_i : W^i \to W^i$ and $I_j : W^j \to W^i$ are the identity maps, with the obvious convention when either i or j is zero. It should be noted that if d is an odd (even) map from W to $T(W)$, the maps d_k, $d_{l,k}$, and $\widehat{d}_k$ are all odd (even) as well. If $d : W \to T(W)$ is any map, then it extends uniquely to a derivation of $T(W)$, so that there is a one-to-one correspondence between derivations on $T(W)$ and maps from W to $T(W)$.

We say that $\widehat{d}$ is a differential if $\widehat{d}^2 = 0$. This condition is also determined by the mapping d alone. Since

(13) $\qquad \widehat{d}^2(a \otimes b) = \widehat{d}(da \otimes b + (-1)^{ad}a \otimes db) = \widehat{d}d(a) \otimes b + a \otimes \widehat{d}db,$

it is clear that the necessary condition $\widehat{d}d = 0$ is sufficient for $\widehat{d}^2 = 0$ as well. Hence,

(14) $$\sum_{k+l=n} d_{k,l}d_l = 0$$

for all $n > 1$. This yields an infinite set of relations which are necessary for $\widehat{d}^2 = 0$, and these are evidently sufficient as well. These relations are simpler than the more complete set of relations $\sum_{k+l=n} \widehat{d}_k\widehat{d}_l = 0$ for all $n > 1$. Since $d_{1,1} = d_1$, the first relation yields $d_1^2 = 0$, so that d_1 determines a differential on W. Clearly, $\widehat{d}_1^2 = 0$, which shows that $\widehat{d}_1$ is itself a differential.

Now the actual object we want to study is the dual of the object that we have been considering. More precisely, let $V = \Pi(W^*)$, where Π denotes the parity reversion of W. (The parity reversion ΠW of a superspace W is given by taking the same underlying space, but assigning opposite parity to each element.) Then the map $\pi : W \to \Pi W$ given by the identity mapping is an odd map. A derivation $\widehat{d}$ induces a dual map $\widehat{m} : T(V) \to T(V)$, which is completely determined by the dual $m : T(V) \to V$ of d. We omit the details of this construction, but note that the signs which occur in the formulas below are obtained from this dualization of the map, using the exchange rule. Similarly, d_k and $d_{l,k}$ induce dual maps

$m_k : V^k \to V$ and $m_{l,k} : V^{k+l-1} \to V^k$. If $\widehat{d}$ is a differential, these maps satisfy the condition

$$(15) \qquad \sum_{k+l=n} m_k m_{l,k} = 0.$$

We can express $m_{l,k}$ in terms of m_l as follows:

$$m_{l,k} = \sum_{i+j=k-1} (-1)^{i(l+1)+m(k-1)} I_i \otimes m_l \otimes I_j,$$

where $m = |\widehat{d}|$ is the parity of m. Notice that for an odd derivation $\widehat{d}$, the associated maps m_l and $m_{l,k}$ are odd for odd l and even for even l, while for $\widehat{d}$ is even the situation is reversed. When $\widehat{d}$ is an odd differential, the condition $\sum_{k+l=n+1} m_k m_{l,k} = 0$ yields the relations

$$(16) \qquad \sum_{\substack{k+l=n+1 \\ i+j=k-1}} (-1)^{s_{i,l}} m_k(v_1, \ldots, v_i, m_l(v_{i+1}, \ldots, v_{i+l}), v_{i+l+1}, \ldots, v_n) = 0,$$

where $s_{i,l} = l(v_1 + \cdots + v_i) + i(l+1) + n - l$. This set of relations defines the structure of a strongly homotopy associative algebra, also called an A_∞ algebra, on V. (This structure was introduced by Stasheff [15, 16].) The consideration above shows that in the case when V is finite-dimensional, an A_∞ algebra structure on V can be defined in terms of operators acting on the tensor algebra $T(W)$ of $W = (\Pi V)^*$. Namely, the A_∞ structure on V can be specified by means of an odd derivation $\widehat{d}$ of this algebra satisfying $\widehat{d}^2 = 0$.[1]

For $n = 1$, the relation (16) becomes $m_1^2(v) = 0$, which means that m_1 is a differential on V. For $n = 2$ the relation becomes

$$(17) \qquad m_1(m_2(a, b)) - m_2(m_1(a), b) - (-1)^a m_2(a, m_1(b)) = 0$$

which says that m_1 is a derivation on the product on V determined by m_2. For $n = 3$, the relation yields

$$(18) \qquad \begin{aligned} m_2(m_2(a, b), c) - m_2(a, m_2(b, c)) &= m_1(m_3(a, b, c)) + m_3(m_1(a), b, c) \\ &\quad + (-1)^a m_3(a, m_1(b), c) + (-1)^{a+b} m_3(a, b, m_3(c)) \end{aligned}$$

which says that m_2 is an associative product up to homotopy, which explains the name given to such a structure.

§5. Deformations of A_∞ algebras

Now we consider the case of an infinitesimal deformation of the operator $\widehat{d}$. Thus we consider a derivation of the form $\widehat{d} + t\widehat{\delta}$, where t is an infinitesimal parameter

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[1]When V is not finite-dimensional, one can give a similar definition of an A_∞ algebra by replacing $T(W)$ with its completion, but it is more convenient to dualize the definition by passing from algebras to coalgebras. Namely, one should introduce a coalgebra structure on $\bigoplus_{k=1}^\infty (\Pi V)^k$, and define an A_∞ algebra by means of a codifferential on this coalgebra, i.e., a coderivation with square zero.

whose parity should be opposite to the parity of $\widehat{\delta}$, so that the resulting operator remains odd. This operator is a differential when $(\widehat{d} + t\widehat{\delta})^2 = 0$, which is precisely the condition $\widehat{d\delta} - (-1)^{\widehat{\delta}}\widehat{\delta d} = 0$, or, in other words, $[\widehat{d}, \widehat{\delta}] = 0$, where $[\cdot, \cdot]$ is the superbracket on the superderivation algebra of $T(W)$. We use the fact that the (super)derivations on $T(W)$ are in one to one correspondence with $\operatorname{Hom}(W, T(W))$ to introduce a differential on $\operatorname{Hom}(W, T(W))$ by $\widehat{D(\delta)} = [\widehat{d}, \widehat{\delta}]$. It is easy to check that $D^2 = 0$. If we denote $\widehat{\rho} = D(\widehat{\delta})$, then we calculate that $\rho_n = \sum_{k+l=n+1} d_{k,l}\delta_l - (-1)^{\widehat{\delta}}\delta_{l,k}d_l$.

Now if $\widehat{\delta}$ is any derivation on $T(W)$, then by the same construction as we used to associate m to $\widehat{d}$, we associate an element μ of $\operatorname{Hom}(T(V), V)$ to $\widehat{\delta}$. We define the parity of the associated map to be the same as the derivation it is associated to, but as a map we note that m_l has parity l, and more generally, the parity of μ_l is $|\delta| + l + 1$. If $\widehat{\nu}$ is the map associated to $\widehat{\rho}$, then we compute that

$$(19) \qquad \nu_n = \sum_{k+l=n+1} m_l\mu_{k,l} - (-1)^{\widehat{\mu}}\mu_l m_{k,l},$$

and this process defines a differential $D(\mu) = \nu$ on $\operatorname{Hom}(T(V), V)$. We use the same notation for the differential on $\operatorname{Hom}(T(V), V)$ as on $\operatorname{Hom}(W, T(W))$. Thus we can consider the homology groups determined by these differentials. These homology groups coincide in our finite-dimensional case. We say that the homology obtained in this manner is the Hochshild cohomology of the A_∞ algebra V, and denote it by $HH(V)$.

For convenience in the formulas to follow, we make the following sign convention:

$$(20) \qquad s_{i,l,\mu,n} = (l + \mu + 1)(v_1 + \cdots + v_i) + (l+1)i + \mu(n-l), \quad (i \geq 1)$$

If $\nu = D(\mu)$, then we have

$$\nu_n(v_1, \ldots, v_n)$$
$$= \sum_{\substack{k+l=n+1 \\ 0 \leq i \leq k-1}} (-1)^{s_{i,l,\mu,n}} m_k(v_1, \ldots, v_i, \mu_l(v_{i+1}, \ldots, v_{i+l}), v_{i+l+1}, \ldots, v_n)$$
$$- \sum_{\substack{k+l=n+1 \\ 0 \leq i \leq l-1}} (-1)^{s_{i,k,m,n}+\mu} \mu_l(v_1, \ldots, v_i, m_k(v_{i+1}, \ldots, v_{i+k}), v_{i+k+1}, \ldots, v_n)$$

The kernel of this differential is the space of all infinitesimal deformations of the A_∞ algebra. If m is the collection of maps $m_k : V^k \to V$, which we call the multiplications in V, and $\mu \in \operatorname{Hom}(T(V), V)$, then μ determines an infinitesimal deformation $m + t\mu$ of m precisely when $D(\mu) = 0$. This interpretation is the same as in the case of an associative algebra. However, when we consider what a trivial deformation is, we note that it no longer is determined by a map $\rho_t : V \to V$, as in the case of associative algebras. Namely, we define a trivial deformation of an A_∞ algebra by means of infinitesimal automorphisms of the tensor algebra $T(W)$, or, equivalently, infinitesimal automorphisms of the cotensor algebra $\bigoplus_{k=1}^\infty (\Pi V)^k$. Such an automorphism has the form $\widehat{\rho}_t = I + t\widehat{\lambda}$, where $\widehat{\lambda}$ is a derivation of $T(W)$, and it is easy to check that the corresponding change of the differential is given by the formula $\widehat{d} \to \widehat{d} + tD(\widehat{\lambda})$. This means that the infinitesimal deformations are classified by the homology $HH(V)$.

§6. Cyclic cohomology of A_∞ algebras

We give a definition of the cyclic cohomology of an A_∞ algebra and prove that the infinitesimal deformations of an A_∞ algebra preserving an invariant inner product are classified by the cyclic cohomology. Suppose that $\widetilde{\mu} \in \mathrm{Hom}(T(V), k)$. Then we define $\widehat{D}(\widetilde{\mu})$ by

(21)
$$\widehat{D}(\widetilde{\mu})_{n+1}(v_1, \ldots, v_{n+1}) = \sum_{\substack{k+l=n+1 \\ 0 \le i \le n}} (-1)^{(v_1+\cdots+v_{[i+l]})(v_{[i+l+1]}+\cdots+v_{n+1})+l(n+1)+ni+\mu}$$
$$\times \widetilde{\mu}_l(m_k(v_{[i+l+1]}, \ldots, v_i), v_{[i+1]}, \ldots, v_{[i+l]})$$

An element $\widetilde{m}$ of $\mathrm{Hom}(T(V), k)$ is said to be cyclic if

(22) $$\widetilde{\mu}_n(v_1, \ldots, v_{n+1}) = (-1)^{n+v_{n+1}(v_1+\cdots+v_n)}\widetilde{\mu}_n(v_{n+1}, v_1, \ldots, v_n).$$

One can check the following fact: *If $\widetilde{\mu}$ is cyclic, then $\widetilde{\nu} = \widehat{D}(\widetilde{\mu})$ is also cyclic.*

The fact that $\widehat{D}$ is a differential on the cyclic elements of $\mathrm{Hom}(T(V), k)$ will follow from the considerations below. We define the cyclic cohomology $HC(V)$ to be the homology determined by this differential.

Suppose that an A_∞ algebra V is equipped with an inner product $\langle \cdot, \cdot \rangle$, and that $\mu \in \mathrm{Hom}(T(V), k)$ is induced by the maps $\mu_k : V^k \to V$. Then $\langle \cdot, \cdot \rangle$ is said to be invariant with respect to μ if the maps $\widetilde{\mu}_k : V^{k+1} \to k$, given by

(23) $$\widetilde{\mu}_k(v_1 \otimes \cdots \otimes v_{k+1}) = \langle \mu_k(v_1 \otimes \cdots \otimes v_k), v_{k+1} \rangle$$

are cyclic.

The inner product induces a map $\mathrm{Hom}(T(V), V) \to \mathrm{Hom}(T(V), k)$, by associating the map $\widetilde{\mu}$ to μ, where $\widetilde{\mu}_k$ is given by formula (23). When this is an isomorphism, we can use it to define a differential $\widetilde{D}$ on $\mathrm{Hom}(T(V), k)$, by the rule $\widetilde{D}(\widetilde{\mu}) = \widetilde{D(\mu)}$.

It follows that $\widetilde{\nu} = \widetilde{D}(\widetilde{\mu})$ is given by the formula

$$\widetilde{\nu}(v_1, \ldots, v_{n+1})$$
$$= \sum_{\substack{k+l=n+1 \\ i \le k-1}} (-1)^{s_{i,l,\mu,n}} \widetilde{m}_k(v_1, \ldots, v_i, \mu_l(v_{i+1}, \ldots, v_{i+l}), v_{i+k+1}, \ldots, v_{n+1})$$
$$+ \sum_{\substack{k+l=n+1 \\ i \le l-1}} (-1)^{s_{i,k,m,n}+\mu+1} \widetilde{\mu}_l(v_1, \ldots, v_i, m_k(v_{i+1}, \ldots, v_{i+k}), v_{i+k+1}, \ldots, v_{n+1}).$$

If we assume temporarily that both m and μ are cyclic with respect to the inner product, then we can express the differential as follows:

(24)
$$\widetilde{D}(\widetilde{\mu})(v_1, \ldots, v_{n+1})$$
$$= \sum_{\substack{k+l=n+1 \\ 0 \le i \le k-1}} (-1)^{(v_1+\cdots+v_{i+l})(v_{i+l+1}+\cdots+v_{n+1})+l(n+1)+ni+\mu}$$
$$\times \widetilde{\mu}_l(m_k(v_{i+l+1}, \ldots, v_{n+1}, v_1, \ldots, v_i), v_{i+1}, \ldots, v_{i+l})$$
$$+ \sum_{\substack{k+l=n+1 \\ 0 \le i \le l-1}} (-1)^{(v_1+\cdots+v_i)(v_{i+1}+\cdots+v_{n+1})+ni+l+\mu}$$
$$\times \widetilde{\mu}_l(m_k(v_{i+1}, \ldots, v_{i+k}), v_{i+k+1}, \ldots, v_{n+1}, v_1, \ldots, v_i)$$

Now we drop the assumption that m is cyclic and define a new operator $\widehat{D}$ on cyclic elements of $\mathrm{Hom}(TV, k)$ by the formula above. This map coincides with the map defined in the beginning of this section. It is straightforward to see that $\widehat{D}$ is a differential.

From the foregoing, we see that deformations of an A_∞ algebra that preserve an invariant inner product are classified by cyclic homology. Trivial deformations are given by infinitesimal automorphisms of the coalgebra associated to V preserving the inner product.

§7. A_∞ deformations of associative algebras

Suppose that V is actually an associative algebra, so that m_2 is the associative product and all other multiplications vanish. Then we can consider the deformations of V into an A_∞ algebra. These deformations are given by the coboundary operator of A_∞ cohomology. Now suppose that μ has only one term μ_k. Comparing the coboundary operator with the Hochschild coboundary operator in the $\mathbb{Z}_2$-graded associative algebra case, one sees that the Hochschild coboundary coincides with the A_∞ coboundary. Therefore, we see that A_∞ deformations of an associative algebra are actually classified by the Hochshild cohomology. In other words, an A_∞ deformation is determined by an element in $\prod_{k=1}^{\infty} HH^k(V)$. Similarly, one sees that deformations of an associative algebra with an (invariant) inner product to an A_∞ algebra with an inner product are given by the cyclic cohomology $\prod_{k=1}^{\infty} HC^k(V)$.

§8. Second Order Deformations

We show that the cohomology ring $HH(V)$ possesses a natural structure of a Lie (super) algebra. To see this, consider the dual picture again, with $\widehat{d}$ being a derivation of the tensor algebra $T(W)$. We saw that elements of $\mathrm{Hom}(W, T(W))$ correspond to derivations of $T(W)$. Thus we have a bracket on $\mathrm{Hom}(W, T(W))$ given by the bracket of derivations. Recall that the differential is given in terms of this bracket by $D(\delta) = [d, \delta\}$. It is easy to check that the bracket descends to a bracket on the cohomology. In our finite-dimensional picture, this induces a bracket on the cohomology $HH(V)$. In the general situation, this is still true, but one uses the the bracket of coderivations to define the bracket structure. Actually, more is true. It is known (see [13, 12]) that the cohomology group of a differential Lie algebra has the natural structure of an L_∞-algebra (strongly homotopy Lie algebra). Applying this statement to the situation above we can provide $HH(V)$ with the structure of an L_∞ algebra. Similar considerations give a Lie algebra structure (and moreover an L_∞ structure) on the cyclic cohomology $HC(V)$ of an A_∞ algebra V with an invariant inner product. It is important to stress that this structure depends on the choice of the inner product.

Now we consider a second order deformation of $\widehat{d}$. It is given by $\widehat{d}_t = d + t\delta + t^2\epsilon$, where we set $t^3 = 0$. We assume here that the parameter t is even, although there exists a more general definition. Then the condition $d_t^2 = 0$ is equivalent to $D(\delta) = 0$ and $[\delta, \delta] = 2D(\epsilon)$. Thus δ is a cocycle. Denote its image in homology, by $\overline{\delta}$. Then the second condition means that $[\overline{\delta}, \overline{\delta}] = 0$, which is a necessary and sufficient condition for extending the first order deformation $d + t\delta$ to a second order one. In other words, $[\overline{\delta}, \overline{\delta}]$ is the first obstruction to extending the infinitesimal deformation to a formal deformation of the algebra.

§9. Ribbon Graphs

Let us consider a ribbon graph with each vertex having at least three edges. By definition, a ribbon graph (fatgraph) is a graph together with a fixed cyclic order of the edges at each vertex. Since all graphs we consider will be ribbon graphs, we will omit the word ribbon from now on. We say that a graph is equipped with a metric if a positive number is assigned to each edge. The set σ_Γ of all metrics on the ribbon graph can be identified with $\mathbb{R}_+^k$, where k is the number of edges in Γ. (Here $\mathbb{R}_+^k$ denotes the subset of $\mathbb{R}^k$ consisting of points with positive coordinates.) Therefore, σ_Γ is topologically a cell. There is a standard construction of a closed surface corresponding to a ribbon graph. If the graph is equipped with a metric then the surface can be provided with a complex structure. Let $\mathcal{R}_{g,n}^{met}$ be the union of the cells σ_Γ, where Γ corresponds to a surface of genus g with n punctures. This set has a natural topology. The limit when the length of one of the edges tends to zero corresponds to the contraction of this edge. It is well known [7, 15] that $\mathcal{R}_{g,n}^{met}$ is topologically equivalent to $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$, where $\mathcal{M}_{g,n}$ is the moduli space of compact complex curves of genus g with n marked points. The decomposition of $\mathcal{R}_{g,n}^{met}$ into the cells σ_Γ is not a cell complex; however, one can define the boundary of a cell σ_Γ in the usual manner and thus define the corresponding homology. This homology is closely related to the homology of $\mathcal{M}_{g,n}$.

To define the homology, we need to examine the complex more carefully. First, we notice that there is an obvious method of deciding when two graphs are equivalent, determining the same cell. We also need a notion of an orientation of the cell corresponding to a graph, which is given by choosing a labeling of the edges and an ordering of the holes in the graph. Then two oriented cells are equivalent if there is a graph equivalence between them such that the orientation induced by the mapping agrees with the chosen orientation of the cell.

It is known that $\mathcal{M}_{g,n}$ has a natural complex structure, so that given a choice of the ordering of the holes in the graph, which determines an orientation of $\mathbb{R}_+^n$, one obtains a canonical orientation on the highest dimensional cells in $\mathcal{R}_{g,n}^{met}$. Note that the highest dimensional cells are those corresponding to trivalent graphs.

In [10], there is a general construction that uses an arbitrary A_∞ algebra to associate a function to oriented graphs. Before considering this general construction, we examine a simplified version, which is related to the construction given in [4].

Suppose that V is an associative (super) algebra with a nondegenerate (super) symmetric bilinear form h. Suppose that $\{e_i\}_{i\in I}$ is a basis of V, and the structure constants m_{ij}^k are given by $e_i e_j = m_{ij}^k e_k$. We use the matrix $h_{ij} = h(e_i, e_j)$, and its inverse h^{ij} to raise and lower the indices. The lower structure constants $m_{ijk} = h(e_i e_j, e_k) = m_{ij}^l h_{kl}$ are cyclically (graded) symmetric, so that $m_{ijk} = (-1)^{e_k(e_i+e_j)} m_{kij}$. Similarly, we note that $h^{ij} = (-1)^{e_i e_j} h^{ji}$. We assign a number to a trivalent graph as follows. To each edge in the graph we associate two indices, one for each vertex, and the tensor h^{ij}, where i, j are the indices associated to the edge. To each vertex we assign the tensor m_{ijk}, where i, j, k are the indices associated to the incident edges, and the order is chosen to be consistent with the cyclic order of the edges at the vertex. We multiply all of these symbols together and sums over repeating indices to obtain a number $Z(\Gamma)$.[2]

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[2]In the graded case, one also picks up a sign in each term depending on the parity of the basis elements, since what we are really doing here is performing a series of graded contractions of a

The resulting function $Z(\Gamma)$ depends only on the genus g and the number n of holes in the graph. This is easy to see, since every trivalent graph with the same number of holes and the same genus can be obtained from one such graph by performing a series of simple transformation, called the fusion move which is illustrated below.
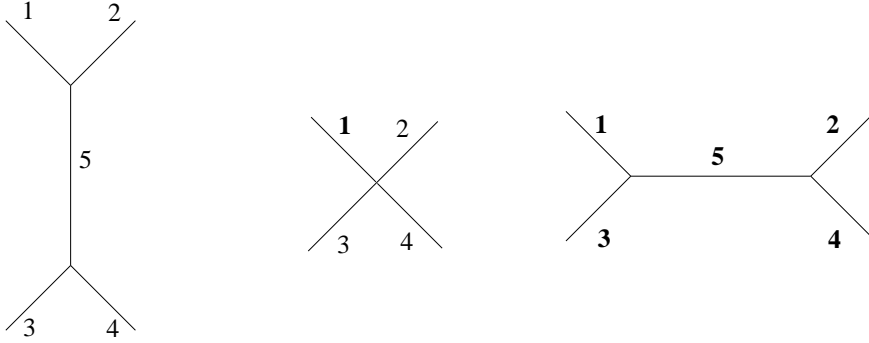

FIGURE 1. Fusion diagram

It is easy to calculate how the functions of two graphs differing by a fusion move are related, and the associativity of the algebra guarantees that the functions will have the same value. Thus if V is equipped with the structure of an associative algebra, we can use the structure constants $m_{ijk} = \widetilde{m}_2(e_i, e_j, e_k)$ to obtain a function that depends only on the genus and number of holes in the graph. We note that in writing down the function $Z(\Gamma)$, an order of the vertices and an order of the edges must be chosen, but the resulting function is independent of these orders because the tensors m_{ijk} and h^{ij} are even, so that the corresponding contractions are the same. We also see that the form of the function $Z(\Gamma)$ depends on the starting edge at each vertex, which determines the order in which the indices for m_{ijk} are listed, but the cyclic symmetry of this tensor again ensures that the function is independent of this choice. Finally, the same arguments show that the function is independent of the order in which the tensor h^{ij} is presented, due to the symmetry of the inner product.

We would like to construct homology classes of the complex $\mathcal{R}_{g,n}^{met}$. With this goal in mind, we construct chains in this complex, that is, linear combinations of oriented cells. Since a change of the orientation corresponds to a change of sign in the complex, we want to define a function on oriented cells that changes sign when the orientation of the cell is reversed. For simplicity, we identify cells with graphs, and therefore replace oriented cells with oriented graphs. Note that this is not the usual notion of an oriented graph. In the discussion above, the function $Z(\Gamma)$ was defined on the set of trivalent graphs. What we really want is to define a function on the set of oriented graphs Γ_{or}, which we shall denote $Z(\Gamma_{\mathrm{or}})$. For an oriented trivalent graph Γ_{or}, we define $Z(\Gamma_{\mathrm{or}}) = Z(\Gamma)$ if the orientation is the canonical one, and $Z(\Gamma_{\mathrm{or}}) = -Z(\Gamma)$ otherwise. It is easy to see that the element

$$(25) \qquad\qquad Z_{\max} = \sum Z(\Gamma_{\mathrm{or}})\Gamma_{\mathrm{or}},$$

tensor according to the prescription which is dictated by the graph.

where the sum is taken over all oriented trivalent graphs, is a cycle in the homology of $\mathcal{R}_{g,n}^{met}$ discussed above, since $Z(\Gamma_{\mathrm{or}})$ is constant on all graphs with the canonical orientation. This result is just the assertion that the cell complex is orientable. Of course, the orientation in this simple case is somewhat superfluous, but in the more general construction to follow, the orientation is quite relevant.

Now suppose that φ_k is a cyclic k-cocycle on the algebra V. We construct a cycle of dimension $k - 2$ less than the maximal dimension on $\mathcal{R}_{g,n}^{met}$. Consider graphs in which all vertices except one are trivalent, and the exceptional vertex has $k + 1$ incident edges. As before, we assign to each trivalent vertex the tensor m_{ijk}, and to the exceptional edge we assign the tensor $\varphi_{i_1 \cdots i_{k+1}} = \widetilde{\varphi}(e_{i_1}, \ldots, e_{i_{k+1}})$. One constructs $Z(\Gamma)$ in the same manner as before, but now there is a problem in the definition if k is odd, because in this case the cyclicity of $\widetilde{\varphi}_k$ means that the formula for $Z(\Gamma)$ depends on which edge one starts with. Of course, the function is determined up to the total sign, since in the expression

$$(26) \qquad \varphi_{i_1 \cdots i_{k+1}} = (-1)^{k + e_{i_{k+1}}(e_{i_1} + \cdots + e_{i_k})} \varphi_{i_{n+1} i_1 \cdots i_k}$$

the sign $(-1)^{e_{i_{k+1}}(e_{i_1} + \cdots + e_{i_k})}$ cancels because this is a graded contraction. Thus only the $(-1)^k$ plays a role. Therefore, in order to assign a fixed value we must consider a starting edge for the vertex.

To explain this, we introduce the notion of a ciliated graph Γ_{cil}, which is a ribbon graph with a preferred edge chosen for each vertex. This terminology was suggested by Fock and Rosly, see [3]. Given a ciliated graph with at most one non trivalent edge, one obtains a well-defined formula for $Z(\Gamma_{\mathrm{cil}})$ by using the preferred edge to determine the order in which to write down the terms for the vertices. Of course, for the trivalent vertices, the choice of the cilia does not affect on the outcome, but for the exceptional vertex the choice is relevant if k is odd. Now we also note that in this case the order of the vertices is not important, because there is at most one vertex corresponding to an odd tensor, so this case is still independent of the order.

We want to define a partition function on oriented graphs that takes opposite signs on graphs of opposite parity. To do this we also use the canonical orientations of trivalent graphs as follows. For the exceptional vertex, we can expand the vertex by inserting edges, using the ciliation as a starting point, to obtain a trivalent graph, as illustrated in the figure below.
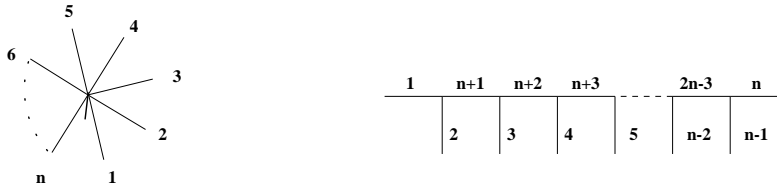

FIGURE 2. Expanding a ciliated vertex

An orientation of the original graph induces an orientation on the trivalent graph, which we can compare to the canonical orientation. If the number of incident edges to the vertex is even, the comparison will depend upon where the cilia is placed, and will alternate as one moves the cilia one edge at a time. On the other hand, if the number of edges is odd, then the orientation of the expanded graph will not depend

on the placement of the cilia. Thus we can define the function $Z(\Gamma_{\mathrm{or}}) = \pm Z(\Gamma_{\mathrm{cil}})$, where the sign is plus if the expanded graph associated to Γ_{or} with the given ciliation has the canonical orientation. Note that result does not depend on the ciliation. One can prove that the element

$$(27) \qquad\qquad Z = \sum Z(\Gamma_{\mathrm{or}})\Gamma_{\mathrm{or}}$$

where the sum runs over all distinct[3] oriented graphs with exactly one nontrivalent vertex with $k+1$ incident edges, is a cycle. We omit a direct proof of this statement. Instead we will derive this result from the fact that a cyclic cocycle determines an infinitesimal deformation of an associative algebra with an invariant inner product into an A_∞ algebra with an invariant inner product. Applying a general result of Kontsevich which we shall explore below, one sees immediately that Z is a cycle.

Now let us consider the case when V is an A_∞ algebra, and graphs with only the restriction that each vertex has at least three edges. We can repeat the construction of the function $Z(\Gamma)$ as before, associating to each vertex with $k+1$ edges the tensor $\widetilde{m}_{i_1 \cdots i_{k+1}}$. The result will depend on the order in which the tensors corresponding to the vertices are listed, more precisely, the order in which the vertices with an even number of incident edges are listed. However, we can resolve this in a similar way by considering graphs with a fixed ciliation and a fixed order of vertices, for which the partition function is well defined, and then multiplying by a sign that depends on the order, ciliation, and orientation, in such a manner that the resulting partition function depends on the orientation alone. To do this, we expand those vertices that are not trivalent, as we did before, but now we must also keep track of the order of the vertices when adding the new labels, so that we not only get an orientation of the graph, but an order of the new holes as well. The canonical orientation is determined by the canonical ordering of the holes as well, so that interchanging the order of the expansion for two vertices will actually produce a reversal of the sign precisely when both vertices are even. But this reversal of sign is mirrored in the contraction as well, so the effect cancels. The ciliation effect is treated as before, so that again we are able to define a function $\mathbb{Z}(\Gamma_{\mathrm{or}})$ depending only on the orientation of the graph.

A result formulated by Kontsevich in this case is that the chain

$$(28) \qquad\qquad Z = \sum Z(\Gamma_{\mathrm{or}})\sigma_{\Gamma_{\mathrm{or}}}$$

is a cycle. To see why this is true, note that it is sufficient to restrict the sum to graphs with a fixed number of edges, and to show that this partial sum is a cycle. Taking a graph with one edge less than this fixed number, we consider the ways to expand this graph by adding one edge. We can consider the graphs obtained by expanding each nontrivalent vertex separately. Restricting attention to those obtained by expanding a single vertex by inserting an edge in one of the possible manners, we obtain expressions for the function that differ only in the contribution from the vertex. If the vertex has n edges, then we will obtain expressions of the form $m_{i_1 \dots i_l} m_{i_{l+1} \dots i_{n+2}}$, multiplied by a sign that is determined by the expansion into a trivalent graph. We will show that the sign coincides with the signs given in the expressions for the relations satisfied by the A_∞ algebra.

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[3]Note that in this construction a graph with opposite orientation is the negative of the oriented graph, so is not considered as distinct in our consideration.

The defining relations for an A_∞ algebra with an invariant inner product can be stated in the form $\widehat{D}(\widetilde{m}) = 0$, from which, using equation 21, we obtain

$$(29) \qquad \sum_{\substack{k+l=n+1 \\ 0 \leq i <= n}} (-1)^{(v_1 + \cdots + v_{[i+l]})(v_{[i+l+1]} + \cdots + v_{n+1}) + l(n+1) + ni + \mu}$$

$$\times \widetilde{m}_l(m_k(v_{[i+l+1]}, \ldots, v_i), v_{[i+1]}, \ldots, v_{[i+l]}) = 0$$

These relations yield the expression

$$(30) \qquad \sum_{\substack{k+l=n+1 \\ 0 \leq i <= n}} -1^s m_{a, j_{[i+1]}, \ldots, j_{[i+l]}} m_{j_{[i+l+1]}, \ldots, j_i, b} = 0$$

where
$$s = (e_1 + \cdots + e_{[i+l]})(e_{[i+l+1]} + \cdots + e_{n+1}) + ni + l + 1.$$

We use these relations to explain the Kontsevich result. We shall concern ourselves here with the factor $ni + l + 1$ which appears in the exponent, as the other part of this exponent is cancelled because this is a graded contraction.

Let us consider a fixed graph, and consider all possible ways to expand this graph into a graph with one additional edge inserted at a fixed vertex. Suppose that this fixed vertex has $n + 1$ edges, where of course, $n \geq 3$. The insertion of an edge will create a graph with two new vertices, having $l + 1$ and $k + 1$ edges, where $k + l = n + 1$. Let us suppose that the original vertex had incident edges $1, \ldots, n + 1$ with associated labels $j_1, \ldots, j_{n+1}$. We insert an edge such a manner that the new graph will will have a vertex with labels a, $j_{[i+1]}$, $\ldots$, $j_{[i+l]}$, and one with labels $j_{[i+l+1]}$, $\ldots$, j_i, b, where a and b are the indices attached to the new edge, which has label $n + 2$. The figure below shows illustrates the labeling of the expanded graph.
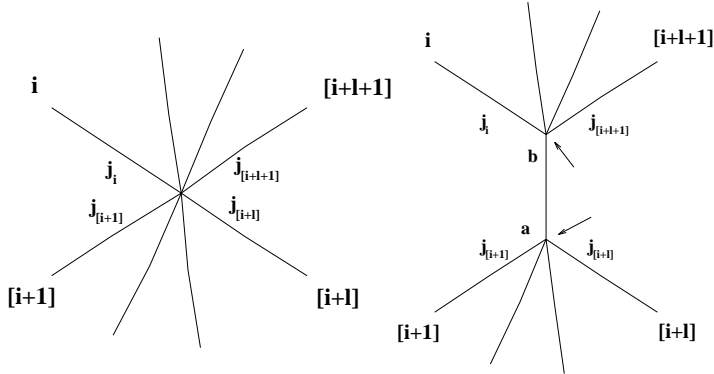

FIGURE 3. Splitting a vertex

Inserting cilia between b and $j_{[i+l+1]}$, and $j_{[i+l]}$ and a respectively, the contribution of the two vertices to the function $Z(\Gamma_{\text{cil}})$ is $m_{a, j_{[i+1]}, \ldots, j_{[i+l]}} m_{j_{[i+l+1]}, \ldots, j_{[i]}, b}$. When expanding the graph according to this ciliation, one sees that the expanded graph is identical to the one which would be obtained from the original graph, with a cilia placed between $j_{[i+l]}$ and $j_{[i+l+1]}$. However, the orientation of the expanded

graph is affected by the fact that the inserted edge has been labeled first, instead of in the order in which it would have been labeled by the usual expansion. In case the original vertex had an odd number of edges, this is the only factor affecting the orientation, because the placement of the cilia in the original vertex is irrelevant. Since n is even, the factor ni does not contribute in the sign $ni + l + 1$. Thus one picks up a factor which depends only on l. On the other hand, when the original vertex had an even number of edges, the factor ni does contribute, which reflects the fact that the placement of the cilia is relevant to the sign of the function $Z(\Gamma_{\mathrm{cil}})$. When we sum over all graphs which result from the insertion of an edge at the fixed vertex, these considerations and equation 30 show that the sum is zero, and this is precisely the incidence number of the graph in the boundary of the chain Z. This shows that this chain is a cycle.

We have tacitly assumed that algebra in question does not have any product of degree one, as the terms involving such a product are not accounted for. However, there are some cases when one can include such a multiplication. We shall not go into this matter here.

The construction of a cycle in $\mathcal{R}_{g,n}^{met}$ by a cyclic cocycle of an associative algebra can be considered as a limiting case of Kontsevich's construction. We can consider this as an infinitesimal deformation of the algebra into an A_∞ algebra. On the other hand, the proof can be carried out directly because the signs in equation 12 coincide with those in equation 30. We can also apply second order deformations in the context above to construct other interesting examples of cocycles.

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